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# FIXED POINT THEOREM IN FUZZY METRIC SPACE USING SUB COMPATIBLE AND SUB SEQUENTIALLY CONTINUOUS MAPS

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**Abstract**

In this paper we prove fixed point theorems in a fuzzy metric space using sub compatible and sub sequentially continuous maps. We offered a generalization of fuzzy metric space. Our results generalize or improve many recent fixed point theorems .some corollaries have been given .At last we give an example to our main results

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**1 INTRODUCTION:**

The concept of Fuzzy sets was introduced by Zadeh [24]. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [14] and George and Veeramani [6] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Vasuki [22] investigated some fixed point theorems in fuzzy metric spaces for R-weakly commuting mappings and pant [15] introduced the notion of reciprocal continuity of mappings in metric spaces. Balasubramaniam, Muralishankar and Pant [15] proved the open problem of Rhoades on the existence of a contractive definition which generals a fixed point but does not force the mapping to be continuous at the fixed point possesses an affirmative answer. In the sequel, Singh and Chauhan [19] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Jain et. al. [10] proved a fixed point theorem for six self maps in a fuzzy metric space and Aage and Salunke[1] also prove a result in this space .Using the concept of compatible maps of type (β), Jain et.al. [10] proved a fixed point theorem in Fuzzy metric space.

In 2009, Al-Thagafi and Shahzad [21] weakened the concept of compatibility by giving a new notion of occasionally weakly compatible (owc) maps which is more general among the commutativity concepts. Bouhadjera and Thobie also prove fixed point theorem for owc maps in weakened the concept of occasionally weak compatibility and reciprocal continuity in the form of sub compatibility and sub sequential continuity respectively and proved some interesting results with these concepts in metric spaces. In 2011, Gopal and Imdad [3] studied the concept of sub-compatible maps in fuzzy metric spaces.

**2. PRELIMINARIES:**

**Definition 2.1** A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is continuous $t - \mathrm{norm}$ if $*$ is satisfying the following conditions:

**2.1** (i) * is commutative and associative.

**2.1** (ii) * is continuous.

**2.1** (iii) $a * 1 = a$ for all $a \in [0,1]$.

**2.1** (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, For $a, b, c, d \in [0,1]$.

**Definition 2.2** A triplet $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t − norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following condition; for all $x, y, z, s, t > 0$,

**2.2** (FM-1) $M(x, y, t) > 0$

**2.2** (FM-2) $M(x, y, t) = 1$ if and only if $x = y$.

**2.2** (FM-3) $M(x, y, t) = M(y, x, t)$

**2.2** (FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

**2.2** (FM-5) $M(x, y, \bullet) : (0,\infty) \rightarrow (0,1]$ is continuous.

Then M is called a fuzzy metric on X. The function $M(x, y, t)$ denote the degree of nearness between x and y with respect to t.

**Example 2.3** Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and

$M(x, y, t) = \frac{t}{t+d(x,y)}$

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For all x, y $\in$ X and all t > 0. Then (X, M, *) is a Fuzzy metric space.

It is called the Fuzzy metric space induced by d.

**Definition 2.4** Two self maps A and B on a fuzzy metric space (X, M, *) are said to be compatible if for all t > 0 ,

$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$

Whenever $\{x_n\}$ is a sequence in X

such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

**Definition 2.5** Two self maps A and B on a fuzzy metric space $(X, M, *)$ are said to be weakly compatible (or coincidently commuting) if they commute at their coincidence points i.e. if $At = Bt$ for some $t \in X$ then $ABt = BAt$.

It is well known fact that compatible maps are weak compatible but the converse is not true.

**Definition 2.6** Two self maps A and B on a set X are said to be owc (occasionally weakly compatible) if and only if there is a point x $\in$X which is a coincidence point of A and B at which A and B commute. i.e., there exists a point $x \in X$ such that $Ax = Bx$ and $ABx = BAx$.

**Definition 2.7** Two self maps A and B on a fuzzy metric space (X ,M , $*$) are said sub compatible if and only if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z,$$

$z \in X$ and which satisfy $\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$ For t > 0.

Obviously two occasionally weakly compatible maps are sub compatible maps, however the converse is not true in general as shown in the following example.

**Example 2.8** Let $X = [0,\infty)$ with usual metric d and define

$M(x, y, t) = \frac{t}{t+d(x,y)}$

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For all x, y $\in$ X and t > 0

Define A, B : X$\to$ X as ;

$$Ax = \begin{cases} x^2, & x < 1 \\ x+4, & x \geq 1 \end{cases}$$

and

$$Bx = \begin{cases} 5x-4, & x < 1 \\ 2x, & x \geq 1 \end{cases}$$

Consider a sequence $x_n = 1 - \frac{1}{n}$ then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} A\left(1 - \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^2 \to 1,$$

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} B\left(1 - \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \left[5\left(1 - \frac{1}{n}\right) - 4\right]$$

$$= \lim_{n\to\infty} \left[1 - \frac{5}{n}\right] \to 1$$

$$\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} A\left(1 - \frac{5}{n}\right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{5}{n}\right)^2$$

$$= \lim_{n\to\infty} \left[1 + \frac{25}{n^2} - \frac{10}{n}\right] \to 1$$

$$\lim_{n\to\infty} BAx_n = \lim_{n\to\infty} B\left(1 - \frac{1}{n}\right)^2$$

$$= \lim_{n\to\infty} \left[5\left(1 - \frac{1}{n}\right)2 - 4\right]$$

$$= \lim_{n\to\infty} \left[5\left\{1 + \left(\frac{1}{n}\right)2 - \frac{2}{n}\right\} - 4\right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{5}{n^2} - \frac{10}{n}\right] \to 1$$

and

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) \to 1$$

Thus, A and B are sub compatible but A and B are not owc maps as,

$$A(4) = 8 = B(4)$$

And

$$AB(4) = A(8) = 12 \neq BA(4) = B(8) = 16.$$

**Definition 2.9** Two self maps A and B on a fuzzy metric space are called reciprocal continuous if

$$\lim_{n\to\infty} Ax_n = At$$

and

$$\lim_{n\to\infty} Bx_n = Bt$$

for some t $\in$ X. whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ax_n = t \text{ for } t \in X .$$

**Definition 2.10** Two self maps A and B on a fuzzy metric space are said to be sub sequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$$

for some t $\in$ X and satisfy

$$\lim_{n\to\infty} Ax_n = At \text{ and } \lim_{n\to\infty} Bx_n = Bt.$$

**Remark 2.11** If A and S both are continuous or reciprocally continuous then they are obviously sub sequentially continuous.

The next example shows that there exist sub sequentially continuous pairs of maps which are neither continuous nor reciprocally continuous.

**Example 2.12** Let X = R , endowed with usual metric d and

$$M(x, y, t) = \frac{t}{t+d(x,y)}$$

For all x, y $\in$ X and $t > 0$. Define $A, B: X \to X$ as ;

$$Ax = \begin{cases} 4, & x < 7 \\ x, & x \geq 7 \end{cases}$$

and

$$Bx = \begin{cases} 4x - 16, & x \leq 7 \\ 7, & x > 7 \end{cases}$$

Consider a sequence $x_n = 7 - \frac{1}{n}$, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} A\left(7 - \frac{1}{n}\right) = 4,$$

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} B\left(7 - \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \left[4\left(7 - \frac{1}{n}\right) - 16\right]$$

$$= \lim_{n\to\infty} \left[4 - \frac{4}{n}\right] = 4$$

$$\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} A\left(4 - \frac{4}{n}\right) = 4 = A(4),$$

$$\lim_{n\to\infty} BAx_n = B(4) = 0 = B(4)$$

Thus A and B are not reciprocally continuous but if we

Consider a sequence $x_n = 7 + \frac{1}{n}$, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} A\left(7 + \frac{1}{n}\right) = 7,$$

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} B\left(7 + \frac{1}{n}\right) = 7$$

$$\lim_{n\to\infty} ABx_n = A(7) = 7 = A(7),$$

$$\lim_{n\to\infty} BAx_n = \lim_{n\to\infty} B\left(7 + \frac{1}{n}\right) = 7 = B(7).$$

Therefore, A and B are sequentially continuous.

## 3. MAIN RESULTS:

**Theorem 3.1:** Let $A, B, S$ and $T$ be four self mappings of a fuzzy metric space $(X, M, *)$. If the pairs $(A, S)$ and $(B, T)$ are sub compatible and sub sequentially continuous, then

**3.1** (I) $A$ and $S$ have a point of coincidence.

**3.1** (II) $B$ and $T$ have a point of coincidence.

Further, if

**3.1**.1(III)

$$\phi[\min\{M(Ax, By, t).M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\}] \geq 0$$

For all $x, y \in X$ and $t > 0$ where $\phi: [0,1] \rightarrow [0,1]$ is a continuous function with $\phi(s) > s$ for each $0 < s < 1$. Then A, B, S and T have a unique common fixed point in X.

**Proof:** Since the pairs (A,S) and (B,T) are sub compatible and sub sequentially continuous, therefore , there are two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u$$

for some $u \in X$ and which satisfy

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = M(Au, Su, t) = 1,$$

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = v$$

for some u $\in$ X and which satisfy

$$\lim_{n\to\infty} M(BTx_n, TBx_n, t) = M(Bv, Tv, t) = 1.$$

Therefore, $Au = Su$ and $Bv = Tv$.

i.e. u is the coincidence point of A and S and v is the coincidence point of B and T.

Now using **3.1** (III) for x= $x_n$ and y = $y_n$ ,we get

$$\Phi[\min\{M(Ax_n, By_n, t).$$
$$M(Sx_n, Ty_n, t), M(Sx_n, Ax_n, t), M(Byn, Ty_n, t)\}] \geq 0$$

On taking limit as $n \rightarrow \infty$

$$\phi[\min\{M(u,v,t)M(u,v,t), M(u,u,t), M(v,v,t)\}) \geq$$
$$\phi(\min\{M(u,v,t),1,1\} \geq 0$$

i.e. $$\phi[\min\{M(u,v,t), M(u,v,t) =\}] > 0$$

which contradiction. Hence u = v.

Again using **3.1** (III) for $x = u, y = y_n$ we obtain

$$\phi[\min\{M(Au, By_n, t)\, M(Su, Ty_n, t), M(Su, Au, t), M(By_n, Ty_n, t)\}] \geq 0$$

On taking limit as $n \to \infty$

$$\phi[\min\{ M(Au,v,t), M(Su,v,t), M(Su,Au,t), M(v,v,t)\} \geq 0]$$

$$\phi[\min\{M(Su,v,t),1,1\}] \geq 0$$

$$\geq \phi[\min\{M(Su,v,t)\}] \geq 0$$

$$\geq \phi[\min\{M(Au,v,t)\}] \geq 0$$

Since $Au = Su$ i.e.

$$\phi[\{M(Au,v,t), M(Au,v,t)\}] \geq 0$$

$$> M(Au,v,t), \geq 0$$

Which yields $Au = v = u$.

Therefore $u = v$ is a common fixed point of A, B, S and T.

**Uniqueness:** Let $w \neq u$ be another fixed point of A, B, S and T.

Then from **3.1** (III) we have

$$\phi[\min\{M(Au,Bw,t), M(Su,Tw,t), M(Su,Au,t), M(Bw,Tw,t)\}] \geq 0$$

$$\phi[\min\{M(Au,Bw,t),1,1\}] \geq 0$$

$$\phi[\min\{M(Au,Bw,t)\}] \geq 0$$

$$\phi[\min\{M(Au,Bw,t)\}] \geq 0$$

$$\phi\ [\{M(Au,Bw,t) M(Au,Bw,t)\}] \geq 0$$

Which yields w = u and therefore uniqueness follows.

If we put A = B and S = T in Theorem – 3.1, we get the following result.

**Corollary 3.2.**Let $A$ and $S$ be two self mappings of a fuzzy metric space

$(X, M, *)$. If the pairs $(A, S)$ is sub compatible and sub sequentially continuous, then

**3.2** (I)A and S have a point of coincidence.

Further, if

**3.2** (II) $\phi[\min\{M(Ax,Ay,t),M(Sx,Sy,t),M(Sx,Ax,t),M(Ay,Sy,t)\}]$
$\geq 0$

For all $x,y \in X$ and $t > 0$ where $\phi$:[0,1] →[0,1] is a continuous function with $\phi(s) > s$ for each $0 < s < 1$.Then A, and S have a unique fixed point in X.

If we put S = T in Theorem- **3.1**, we get the following result.

**Corollary 3.3** Let $A, B$ and $S$ be three self mappings of a fuzzy metric space $(X, M, *)$. If the pairs $(A, S)$ and $(B, S)$ are sub compatible and sub sequentially continuous, then

**3.3** (I) $A$ and $S$ have a point of coincidence.

**3.3** (II) $B$ and $S$ have a point of coincidence.

further, if

**3.3** (III) $\phi[\min\{M(Ax,By,t),M(Sx,Sy,t),M(Sx,Ax,t),M(By,Sy,t)\}] \geq 0$

for all $x,y \in X$ and $t > 0$ where $\phi: [0,1] \rightarrow [0,1]$ is a continuous function with $\phi(s) > s$ for each $0 < s < 1$.Then A, B and S have a unique fixed point in X.

Now, we furnish our theorem with example.

**Example 3.4:** Let X = R, equipped with usual metric d and

M(x, y, t) = $\frac{t}{t+d(x,y)}$

For all x, y $\in$ X and all t > 0.

Define the maps A, B, S and $T: X \rightarrow X$ as

Ax = $\begin{cases} 0 & ,x \leq 0 \\ 3x+1 & ,x > 0 \end{cases}$ Bx = $\begin{cases} 2x, & x \leq 0 \\ 5 & ,x > 0 \end{cases}$

Sx = $\begin{cases} x^2 & ,x \leq 0 \\ 3x-1 & ,x > 0 \end{cases}$ Tx = $\begin{cases} x-2, & x < 0 \\ x^2-x & ,x \geq 0 \end{cases}$

Consider the sequences $\{x_n\} = \{y_n\} = \frac{1}{n}$

Then, clearly $Ax_n$ , $Bx_n$, $Sx_n$ and $Tx_n \rightarrow 0$

$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(\frac{1}{n}\right) \rightarrow 0 = A(0)$ and

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{n}\right)^2 \rightarrow 0 = S(0)$$

Thus $(A, S)$ is sub compatible and sub sequentially continuous.

Again, $\lim_{n\to\infty} BT(x_n) = \lim_{n\to\infty} B\left(\frac{1}{n}\right) = \lim_{n\to\infty} 2\left(\frac{1}{n}\right) \rightarrow 0 = B(0)$ and

$\lim_{n\to\infty} TB(x_n) = \lim_{n\to\infty} T\left(\frac{1}{n}\right) = \lim_{n\to\infty} \left[\left(\frac{1}{n}\right)^2 - \frac{1}{n}\right] \rightarrow 0 = T(0)$

Which shows that (B , T ) is sub compatible and sub sequentially continuous.

Also the condition 3.1(III) of our Theorem is satisfied and 0 is unique common fixed point of A, B, S and T.

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